Chapter One: Functions

- **Relations**

***Definition* 1.1:**

Let A and B be two non-empty sets, A relation R from a set A into a set B is a subset from the cross product $A\times B$. Where

$$A\times B=\{(a,b)|\ a\in A \text{ and } b\in B\}$$

In such a relation R above, the element b is called the image of the element a.

***Example* 1.2:**

Let $A=\{a,b,c\}$ and $B=\{1,2\}$.

$R_1=\{(a,1),(a,2),(b,1)\}$ is a relation.

$R_2=\{(a,1),(b,2)\}$ is a relation.

$R_3=\{(a,1),(a,2),(b,3)\}$ is not a relation from A to B since $3\notin B$.

$R_4=\{(b,2)\}$ is a relation.

$R_5=A\times B$ is a relation.

- **Functions**

***Definition* 1.3:**

Let A and B be two non-empty sets. A function f from A into B is a relation from A into B such that every element $a\in A$ has a unique (one and only one) image $b\in B$.

We will use the notation $f{:}A\rightarrow B$ to denote a function from A to B and $b=f(a)$ instead of $(a, b)\in f$.

***Definition* 1.4:**

Let $f{:}A\to B$ be a function, the set A is called the domain of f and denoted by D_f and the set B is called the codomain of f. In other words the domain is the set of all points that makes the function f to be defined.

The range of a function f denoted by R_f and defined as follows:

$$R_f=\{b\in B|\text{ there exists } a\in A \text{ such that } b=f(a)\}.$$

***Definition* 1.5**

A function $f{:}A\to B$ is said to be a real-valued if B is a subset of the set of real numbers.

In this course, we will assume only the real-valued functions $y=f(x)$. x is called the independent variable and y is called the dependent variable.

Polynomials

***Definition* 1.6**

A polynomial in a one variable x is a function of the form

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0,$$

where a_i, $i=0,1,\ldots,n$ are real numbers called the coefficients and n is a non-negative integer called the degree of the polynomial p.

- ***How do we find the domain and the range of functions* (*real-valued functions)***

1. Rational Functions $f(x)=\frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials

For this type of functions, we will exclude the domain from the elements that make $h(x)=0$.

To find the range of f, we will rewrite y as a function of the variable x and follow the same instruction of as we do for finding the domain.

***Example* 1.7**

Find the domain of the function

$$f(x) = \frac{x+2}{x^2 - 5x - 14}$$

***Example* 1.8**

Find the domain of the function

$$f(x) = \frac{x}{|x|}$$

2. *n*-th root function $\sqrt[n]{polynomial}$, where n is an even integer

   For this type of functions, if n is even $\{2,4,6,\ldots\}$, then the domain is the set of all values that make the *polynomial* to be non-negative. Otherwise, the domain is the set of all real numbers.

   ***Example* 1.9**

   Find the domain of each one of the following

   1. $f(x)=\sqrt{x}$
   2. $g(x)=\sqrt{x-4}$
   3. $h(x)=\sqrt{2-x}$
   4. $f(x)=\sqrt{x^2-9}$
   5. $f(x)=\sqrt{16-x^2}$
   6. $f(x)=\sqrt{x^2+2}$
   7. $f(x)=\sqrt[3]{x}$

***Example* 1.10**

Find the domain of the function

$f\,(x)=\frac{x-2}{\sqrt{3-x^2}}$

***Exercises* 1.11**

Find the domain of the following functions

1. $f\,(x)=\frac{3}{x^2-4x+4}$
2. $f\,(x)=\sqrt{x^2+2x-15}$
3. $f\,(x)=\sqrt{6-x}$
4. $f\,(x)=\frac{x^2-x-2}{\sqrt{x^2-25}}$

Graph of Functions

The graph of a function $y=f\,(x)$ is the set

$$G=\{(x,y)|x\in D_f\}$$

***Example* 1.12**

Sketch the graph of the following functions

1. $f\,(x)=x$
2. $f\,(x)=-x$
3. $f\,(x)=x^2$
4. $f\,(x)=-x^2$
5. $f\,(x)=|x|$

***How to draw polynomial functions up to degree* 2**

To do that, we assume the above functions in ***Example* 1.12** as parent functions of degree 1 and 2.

***Example* 1.13**

Sketch the following functions

1. $f(x)=x-2$
2. $f(x)=3-x$
3. $f(x)=(x-1)^2$
4. $f(x)=-(x+3)^2$
5. $f(x)=-x^2-2$
6. $f(x)=x^2+3$
7. $f(x)=x^2-6x+13$
8. $f(x)=|x-4|$
9. $f(x)=|2x-1|$
10. $f(x)=-|x|$

***Example* 1.14**

Sketch the graph of

1. $f(x)=\sqrt{x}$
2. $f(x)=\sqrt{x-2}$

3. $f(x)=\sqrt{x}+1$
4. $f(x)=\sqrt{x-1}+2$
5. $f(x)=-\sqrt{x}$
6. $f(x)=\sqrt{-x}$

***Example* 1.15**

Sketch the graph of the following functions

1. $f(x)=x^3$
2. $f(x)=x^4$

***Exercises* 1.16**

Sketch the graph of the following functions

1. $f(x)=x^2+7$
2. $f(x)=2x+8-x^2$
3. $f(x)=2-\sqrt{x+3}$
4. $f(x)=\sqrt{-x-1}$
5. $f(x)=(x-2)^3+1$
6. $f(x)=|x+4|$

Even and odd functions

***Definition* 1.17**

A function $y=f(x)$ is called

1. even if $f(-x)=f(x)$
2. odd if $f(-x)=-f(x)$

***Example* 1.18**

1. $f(x)=x^2$ is an even function.
2. $f(x)=x^3-x$ is an odd function.
3. $f(x)=x+1$ is neither even nor odd function

***Remark* 1.19**

The graph of even functions is symmetric with the y-axis, while the graph of odd functions is symmetric with the origin.

Increasing and Decreasing functions

Let $y=f(x)$ be a function defined on an interval I (closed or open or half (right or left) open interval). A function f is said to be

1. Increasing if $f(x_1)\leq f(x_2)$, where $x_1<x_2$.
2. Decreasing if $f(x_1)\leq f(x_2)$, where $x_1>x_2$.

***Example* 1.20**

1. The function $y=x^2$ is decreasing in the interval $]-\infty,0)$ and increasing in the interval $(0,\infty[$.
2. The function $y=x^3$ is increasing on $]-\infty,\infty[$.
3. The function $y=f(x)=2$ is neither increasing nor decreasing.

***Exercises* 1.21**

A. Determine whether the following functions even or odd or non.

1. $f(x)=x^4-x^2$
2. $f(x)=|x|$
3. $f(x)=x+7$
4. $f(x)=\sqrt{x+2}$

5. $f(x)=\frac{x^3}{x}$
6. $f(x)=x^5-x^3-x+2$

B. Determine whether the following functions increase or decrease or non in their corresponding intervals

1. $f(x)=|x+2|$, in the interval (-4,4].
2. $f(x)=\sqrt{4-x^2}$, in the interval [-2,2].
3. $f(x)=x$, in the interval (0,5).
4. $f(x)=-\sqrt{-x}$, in the interval $]-\infty,0]$.

Floor and Ceiling function

***Definition* 1.22**

1. The floor function is a function denoted by $f(x)=\lfloor x\rfloor$ and defined as follows: For a real number x, $\lfloor x\rfloor$ is the nearest integer less than x. If x is an integer, then $\lfloor x\rfloor=x$.
2. The ceiling function is a function denoted by $g(x)=\lceil x\rceil$ and defined as follows:

   For a real number x, $\lfloor x\rfloor$ is the nearest integer greater than x. If x is an integer, then $\lceil x\rceil=x$.

***Example* 1.23**

$f(1.2)=\lfloor 1.2\rfloor=1$ and $g(1.2)=\lceil 1.2\rceil=2$, while $f(2)=\lfloor x\rfloor=2=\lceil 2\rceil=g(2)$.

***Remark* 1.24**

The range of both floor and ceiling functions are the set of integer numbers.

***Question*:**

Sketch the graph of floor and ceiling functions.

***Example* 1.25**

Solve the following equations

1. $\lfloor x-1 \rfloor=0$.
2. $\lceil x-1 \rceil=0$.
3. $\lfloor x \rfloor^2-8\lfloor x \rfloor-2=-17$.
4. $2\lceil x \rceil-3=0$.

***Exercises* 1.26**

A. Find the domain and the range of the following functions

1. $f(x)=\sqrt{x-\lfloor x \rfloor}$
2. $f(x)=\sqrt{x-\lceil x \rceil}$

B. Solve the following equations

1. $\lceil x+1 \rceil^2-9=0$.
2. $(2+x)\lfloor x \rfloor=0$.

System of Linear Equations with Two Variables

***Definition* 1.27**

Assume the following two dimensional linear system

$$\begin{aligned} ax + by &= e_1 \\ cx + dy &= e_2 \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The system (1) is called consistent if it is either independent or dependent. By independent system, we mean a system having exactly one solution (x,y), and dependent if it has infinite number of solutions.

The system (1) is called inconsistent if it is not consistent, in other words, the system has no solution.

***Example* 1.28**

Solve the following systems

1. $\begin{aligned} 2x + 3y &= -4 \\ 3x - 5y &= 14 \end{aligned}$

2. $\begin{aligned} 3y - x &= 1 \\ 4x + 4 &= 12y \end{aligned}$

3. $\begin{aligned} 3x - 5y &= 21 \\ 15y &= 14 + 9x \end{aligned}$

***Exercise* 1.29**

Solve the following systems

1. $\begin{aligned} \frac{3}{4}x - 2y &= 2 \\ \frac{-1}{2}x + \frac{2}{5}y &= 14 \end{aligned}$

2. $\frac{2}{3}y - 6 = -x$

$\frac{-1}{2}x + 3 = \frac{1}{3}y$

3. $2x + \frac{2}{3}y = -\frac{7}{2}$

$\frac{2}{9}y = \frac{16}{3} - \frac{2}{3}x$